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# DILATATION ESTIMATES FOR QUASICONFORMAL EXTENSIONS[†]

BY

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ABSTRACT

Let $D$ and $D'$ be ring domains in $B^n$, with $S^{n-1}$ as one boundary component, and let $f:\bar{D}\to\bar{D}'$ be a homeomorphism which is $K$-quasiconformal in $D$ and with $f(S^{n-1})=S^{n-1}$. According to a result of Gehring $f\mid S^{n-1}$ admits an extension $g:\bar{B}^n\to\bar{B}^n$ which is quasiconformal in $B^n$. We find here an upper bound for the dilatation of $g$ in terms of $n$, $K$, and $\operatorname{mod} D$.

## 1. Introduction

1.1. It is known that the following extension theorem holds in $\bar{R}^n$ (theor. 1, [2]):

THEOREM 1. *Suppose that $D_1$ and $D_2$ are Jordan domains in $\bar{R}^n$ with $\bar{D}_1\cap\bar{D}_2=\emptyset$ and that $B_1$ and $B_2$ are open balls in $R^n$ with $\bar{B}_1\cap\bar{B}_2=\emptyset$. Suppose also that $f$ is a homeomorphism of $C(D_1\cup D_2)$ onto $C(B_1\cup B_2)$, that $f$ is quasiconformal in $C(\bar{D}_1\cup\bar{D}_2)$ and that $f(\partial D_i)=\partial B_i$ for $i=1,2$. Then there exists a homeomorphism $f^*$ of $C(D_2)$ onto $C(B_2)$ such that $f^*$ is quasiconformal in $C(\bar{D}_2)$ and $f^*=f$ in $\partial D_2$.*

We are concerned with finding an upper bound for the maximal dilatation $K^*$ of $f^*$ under the additional hypothesis that $D_2$ is an open ball. The method is to reduce the situation by auxiliary conformal mappings to the case where lemma 2 [2] applies and to estimate the maximal dilatations of the auxiliary mappings $h$ of lemma 1 [2] and $g$ of lemma 2 [2].

1.2. *The results.* With the notation of Theorem 1, let $A=C(\bar{D}_1\cup\bar{D}_2)$, $m=\operatorname{mod} A$, and let $K(f)$ be the maximal dilatation of $f$. Let

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$$F(n, K, m) = \{f \mid f \text{ as in Theorem 1, } K(f) \leqq K\}.$$

For $f \in F(n, K, m)$ denote

$$F^*(f) = \{f^* \mid f^* : CD_2 \to CB_2 \text{ homeomorphism, quasiformal in } C\bar{D}_2, \quad f^* = f \text{ in } \partial D_2\},$$

$$\varphi(n, K, m) = \sup\{\inf\{K(f^*) \mid f^* \in F^*(f)\} \mid f \in F(n, K, m)\}.$$

If $D_2$ is an open ball, we show in (5.9) that for every $f \in F(n, K, m)$ there exists an $f^* \in F^*(f)$ with

$$K(f^*) \leqq C_1(n, K) C_2(n, K)^{\exp(2 \operatorname{mod} A)}(1 + K^{2n} \operatorname{mod} A^{-2n}),$$

where $C_2(n, K) > 1$. Since the upper bound only depends on $n$, $K$, and $m = \operatorname{mod} A$, it is also an upper bound for $\varphi(n, K, m)$. By (5.10), the upper bound grows as $\operatorname{mod} A \to 0$ like

$$C_3(n, K) \operatorname{mod} A^{-2n}.$$

In (5.13) and (5.14) this is compared with an example of a function with large maximal dilatation for $n = 2$, $K = 1$.

$\varphi$ is a decreasing function of $m = \operatorname{mod} A$. This is seen as follows: Let $m' < m$, $f \in F(n, K, m)$ and $f'$ be a restriction of $f$ into a ring domain $A' = C(\bar{D}'_1 \cup \bar{D}_2)$ with $\operatorname{mod} A' = m'$. The modulus $m'$ is attained because of the continuity property of the modulus [1]. Then $F^*(f) = F^*(f')$, hence $\varphi(n, K, m) \leqq \varphi(n, K, m')$.

It would seem natural to conjecture $\lim \varphi(n, K, m) = K$ as $m \to \infty$.

## 2. Terminology

We let $R^n$ denote the $n$-dimensional euclidean space, $n \geqq 2$, and $\bar{R}^n$ its one point compactification. For each set $E \subset \bar{R}^n$ we let $\partial E$, $\bar{E}$ and $CE$ denote the boundary, closure, and complement of $E$ in $\bar{R}^n$. Finally for $E$, $F \subset \bar{R}^n$ we let $d(E, F)$ denote the euclidean distance between $E$ and $F$, and $\operatorname{diam}(E)$ the diameter of $E$.

A homeomorphism $f$ of a domain $D \subset \bar{R}^n$ is said to be *$K$-quasiconformal*, $1 \leqq K < \infty$ if it satisfies the double inequality

$$M(\Gamma)/K \leqq M(f(\Gamma)) \leqq KM(\Gamma)$$

for each family of curves $\Gamma$ in $D$, where $M(\Gamma)$ is the $n$-modulus of $\Gamma$, see [6, 6.1]. The maximal dilatation of $f$ is defined as the least $K$ for which $f$ is $K$-quasiconformal.

We denote $B^n(x_0, r) = \{x \in R^n \mid |x - x_0| < r\}$, $S^{n-1}(x_0, r) = \{x \in R^n \mid |x - x_0| = r\}$, $S^{n-1}(r) = S^{n-1}(0, r)$, $S^{n-1} = S^{n-1}(0, 1)$ and $\omega_{n-1} = m_{n-1}(S^{n-1})$.

For definitions of a *ring domain, condenser, modulus of a ring domain, capacity of a condenser*, and their relations with modulus of a curve family, see [4] and [7].

## 3. Construction

3.1. *Normalization.* We start by normalizing the situation in the following manner: Let $\mathcal{A}$, $\mathcal{A}' \subset \bar{R}^n$ be ring domains and $F: \bar{\mathcal{A}} \to \bar{\mathcal{A}}'$ be a homeomorphism such that $F \mid \mathcal{A}$ is $K$-quasiconformal. Denote the components of the complement of $\mathcal{A}$ by $\bar{\mathcal{D}}_1$, $\bar{\mathcal{D}}_2$ and those of $\mathcal{A}'$ by $\bar{\mathcal{B}}_1$, $\bar{\mathcal{B}}_2$. Suppose that $\mathcal{D}_2$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are balls and $\mathcal{D}_1$ a Jordan domain. By using auxiliary conformal mappings we can assume that $\mathcal{D}_2$ and $\mathcal{B}_2$ are lower half-spaces. Suppose further that $F(\partial\mathcal{D}_i) = \partial\mathcal{B}_i$ for $i = 1, 2$. Denote $b = \mathrm{d}(\partial\mathcal{D}_1, \partial\mathcal{D}_2)$, $d = \mathrm{diam}(\mathcal{D}_1)$.

3.2. *Construction.* Let $y_1 \in \partial\mathcal{D}_1$ be a point with maximal distance from $\partial\mathcal{D}_2$ (Fig. 1). We choose a point $Q_R \in \mathcal{D}_2$ on the line $\rho$ through $y_1$, normal to $\partial\mathcal{D}_2$, such that $R = \mathrm{d}(Q_R, \partial\mathcal{D}_2) = b + d$ and make a reflection $I_1$ in $S^{n-1}(Q_R, R)$. Let

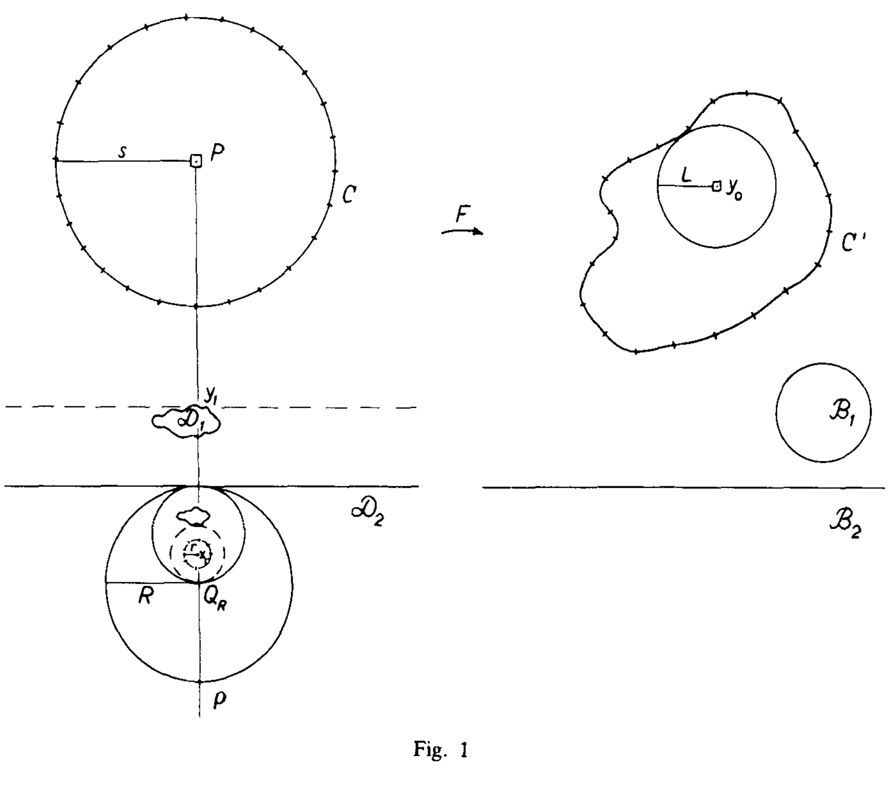


Fig. 1

$B^n(x_0, h/2)$ be the biggest ball in $I_1(\mathscr{A})$ such that $x_0 \in \rho$ and $Q_R \in S^{n-1}(x_0, h/2)$. Then $h = \mathrm{d}(I_1(y_1), Q_R)$ and we have, by denoting $h = 4r$,

$$4r = h \geqq R^2/(R + b + d) = R/2. \tag{3.3}$$

Next make a reflection $I_2$ in $S^{n-1}(x_0, r)$, followed by the normalization $N_1: x \mapsto (x - x_0)/r$. Denote $N_1 \circ I_2 \circ I_1(\mathscr{D}_i) = D_i$, $i = 1, 2$. Let $a = \mathrm{d}(0, \partial D_2)$, $a_1 = \max\{|y| : y \in \partial D_1\}$, $a_2 = \max\{|y| : y \in \partial D_2\}$. By (3.3)

$$a = (R/r - 2)^{-1} \geqq 1/6, \qquad a_1 = 1/2 = a_2. \tag{3.4}$$

Hence, the maximal dilatation $K_3$ of the mapping $g$ of lemma 2 [2] has an upper bound

$$K_3 = (\log a/\log a_2)^{n-1} \leqq (\log 6/\log 2)^{n-1} < 3^{n-1}. \tag{3.5}$$

For the maximal dilatation $K_2$ of the mapping $h$ of lemma 1 [2] we still have to deform the image with two conformal mappings.

Consider the set $C = (N_1 \circ I_2 \circ I_1)^{-1} S^{n-1}$. It is a sphere in the upper half-space $H$. Denote $C = S^{n-1}(P, s)$. Then $F(C) = C' \subset H$. Let $y_0 = F(P)$, $L = \mathrm{d}(y_0, C')$, and $I_3$ be the reflection in $S^{n-1}(y_0, L)$. Denote $I_3(\mathscr{B}_i) = B^n(P_i, r_i) = B_i$, $i = 1, 2$, and let $t = \mathrm{d}(B_1, B_2)$. Let $N_2$ be the normalization $x \mapsto (x - y_0)/(L(1 + \varepsilon))$ with $\varepsilon > 0$ and let $f = N_2 \circ I_3 \circ F \circ (N_1 \circ I_2 \circ I_1)^{-1}$. Then $f: \mathrm{C}(D_1 \cup D_2) \to \mathrm{C}(N_2(B_1) \cup N_2(B_2))$ fulfills the conditions of lemma 2 [2], and the formula for $K_2$ of lemma 1 [2] is

$$K_2 = (1 + 9(1 + \varepsilon)L/t)^n. \tag{3.6}$$

## 4. Estimation of $K_2$

4.1. To estimate $K_2$ we need a positive lower bound for $t/L$. Denote $C'' = I_3(C')$ and $l = \mathrm{d}(P_2, C'')$. We write the identity

$$\frac{t}{L} = \frac{t}{\min(r_1, r_2)} \frac{\min(r_1, r_2)}{l} \frac{l}{L}$$

and find positive lower bounds $c_i$, $i = 1, \cdots, 4$ (which depend only on $n$, $K$, and $\mathrm{mod}\,\mathscr{A}$) for $t/\min(r_1, r_2)$, $l/L$, $r_1/l$ and $r_2/l$, respectively. Then by (3.6)

$$K_2 \leqq (1 + 9(1 + \varepsilon)/(c_1 c_2 \min(c_3, c_4)))^n. \tag{4.2}$$

4.3. *The constant* $c_1$. We derive an estimate for $c_1$ such that $t/\min(r_1, r_2) \geqq c_1$. Let $r = \min(r_1, r_2)$ and let $G$ be the ring domain with two balls of radius $r$ at a distance $t$ as components of the complement. Then $\mathrm{mod}\, G$ is obtained by mapping $G$ conformally onto a concentric spherical ring and

$$\operatorname{mod}\mathcal{A}/K^{1/(n-1)} \leqq \operatorname{mod}\mathcal{A}' \leqq \operatorname{mod} G = \log(a + (a^2 - 1)^{1/2}),$$

where $a = x^2/2 + 2x + 1$ and $t/r = x$. Hence $t/r \geqq \min(1, c^2)$, where $c = (\exp(\operatorname{mod}\mathcal{A}/K) - 1)/8$ and we can choose

$$c_1 = \min(1, c^2). \tag{4.4}$$

4.5. In order to estimate $c_2$, we derive an auxiliary result which is based on the following Lemma 1, which follows from lemmas 1 and 2 in [3]:

LEMMA 1. *Suppose that $B \subset R^n$ is an open ball, that $E$ and $F$ are disjoint compact sets in $CB$ and that $\tilde{E}$ and $\tilde{F}$ are the reflections of $E$ and $F$ in $\partial B$. If $\Gamma$ and $\tilde{\Gamma}$ are the families of curves joining $E$ to $F$ in $C\bar{B}$ (or $CB$) and $E \cup \tilde{E}$ to $F \cup \tilde{F}$ in $\bar{R}^n$, respectively, then*

$$M(\tilde{\Gamma}) = 2M(\Gamma).$$

4.6. LEMMA. *Let $B \subset R^n$ be an open ball with center $P$, $E$ a compact set in $CB$, $\tilde{E}$ the reflection of $E$ in $\partial B$ and $E^*$ and $\tilde{E}^*$ the spherical symmetrizations* [5, 2.7] *of $E$ and $\tilde{E}$, respectively, with respect to $\{P + \lambda e_n \mid \lambda \leqq 0\}$. Then $\tilde{E}^*$ is the reflection in $\partial B$ of $E^*$.*

PROOF. Suppose $B = B^n(0, s)$ and let $t \geqq s$. Considering the following four cases we verify that $\tilde{E}^*$ is the reflection in $\partial B$ of $E^*$.

1. The case $S^{n-1}(t) \cap E$ is null is equivalent to $S^{n-1}(t) \cap E^*$ is null, $S^{n-1}(s^2/t) \cap \tilde{E}$ is null and $S^{n-1}(s^2/t) \cap \tilde{E}^*$ is null.

2. $S^{n-1}(t) \subset E$ is equivalent to $S^{n-1}(t) \subset E^*$, $S^{n-1}(s^2/t) \subset \tilde{E}$ and $S^{n-1}(s^2/t) \subset \tilde{E}^*$.

3. Suppose $S^{n-1}(t)$ meets $E$ in a set whose $(n-1)$-measure $A = 0$. Then $E^*$ meets $S^{n-1}(t)$ exactly at the point $(0, \cdots, 0, -t)$ and $\tilde{E}^*$ meets $S^{n-1}(s^2/t)$ exactly at the point $(0, \cdots, 0, -s^2/t)$.

4. Let $A$ be defined as in 3 and $0 < A < \omega_{n-1}$. Then $E^*$ meets $S^{n-1}(t)$ in a closed spherical cap of $(n-1)$-measure $A$ with center at $(0, \cdots, 0, -t)$. Because $\tilde{E}$ is the reflection of $E$ in $\partial B$, $\tilde{E}$ meets $S^{n-1}(s^2/t)$ in a set whose $(n-1)$-measure is $(s^2/t^2)^{n-1}A$. Hence $\tilde{E}^*$ meets $S^{n-1}(s^2/t)$ in a closed spherical cap of $(n-1)$-measure $(s^2/t^2)^{n-1}A$, with center at $(0, \cdots, 0, -s^2/t)$.

4.7. LEMMA. *Let $B \subset R^n$ be an open ball with center 0, $E_1$ and $E_2$ compact sets in $CB$ and $\Gamma_2$ the family of curves joining $E_1$ to $E_2$ in $CB$. Let $E_1^*$ be the spherical symmetrization of $E_1$ with respect to $\{\lambda e_n \mid \lambda \leqq 0\}$, $E_2^*$ the reflection in the plane $x_n = 0$ of the spherical symmetrization of $E_2$ and $\Gamma_2^*$ the family of curves joining $E_1^*$ to $E_2^*$ outside $B$. Then*

$$M(\Gamma_2) \geqq M(\Gamma_2^*).$$

PROOF. Denote by $\tilde{E}_i$ the reflection of $E_i$ in $\partial B$, $i = 1, 2$, and let $\tilde{\Gamma}_2$ be the family of curves joining $E_1 \cup \tilde{E}_1$ to $E_2 \cup \tilde{E}_2$ in $\bar{R}^n$. By Lemma 1 of 4.5

$$M(\tilde{\Gamma}_2) = 2M(\Gamma_2). \tag{4.8}$$

Next make a spherical symmetrization of the condenser $(C(E_2 \cup \tilde{E}_2), E_1 \cup \tilde{E}_1)$ with respect to $\{\lambda e_n \mid \lambda \leqq 0\}$. Denote by $E_1^*$, $\tilde{E}_1^*$ the symmetrizations of $E_1$, $\tilde{E}_1$ and by $E_2^*$, $\tilde{E}_2^*$ the components of the complement of the spherical symmetrization of $C(E_2 \cup \tilde{E}_2)$. Then $E_2^*$ and $\tilde{E}_2^*$ are the reflections in the plane $x_n = 0$ of the spherical symmetrizations of $E_2$ and $\tilde{E}_2$, respectively. Let $\Gamma_2^*$ be the family of curves joining $E_1^*$ to $E_2^*$ outside $B$ and $\tilde{\Gamma}_2^*$ $E_1^* \cup \tilde{E}_1^*$ to $E_2^* \cup \tilde{E}_2^*$ in $\bar{R}^n$. By Lemma 4.6 $\tilde{E}_i^*$ is the reflection in $\partial B$ of $E_i^*$, $i = 1, 2$, and by Lemma 1 of 4.5

$$M(\tilde{\Gamma}_2^*) = 2M(\Gamma_2^*).$$

Because of symmetrization, [5, 7.5]

$$M(\tilde{\Gamma}_2) \geqq M(\tilde{\Gamma}_2^*).$$

Combining these inequalities with (4.8) we obtain the result.

4.9. *The constant* $c_2$. We want to find $c_2$ such that $l/L \geqq c_2$, where $L = \max|y - y_0|$, $l = \min|P_2 - y|$, $y \in C''$ (Fig. 2). Let $\Gamma'$ be the family of curves joining $S^{n-1}(P_2, l)$ and $S^{n-1}(y_0, L)$ outside $B_1$ and $\Gamma$ its image under $f^{-1} \circ N_2$. If $\Gamma_1'$ joins $S^{n-1}(P_2, l)$ and $S^{n-1}(y_0, L)$,

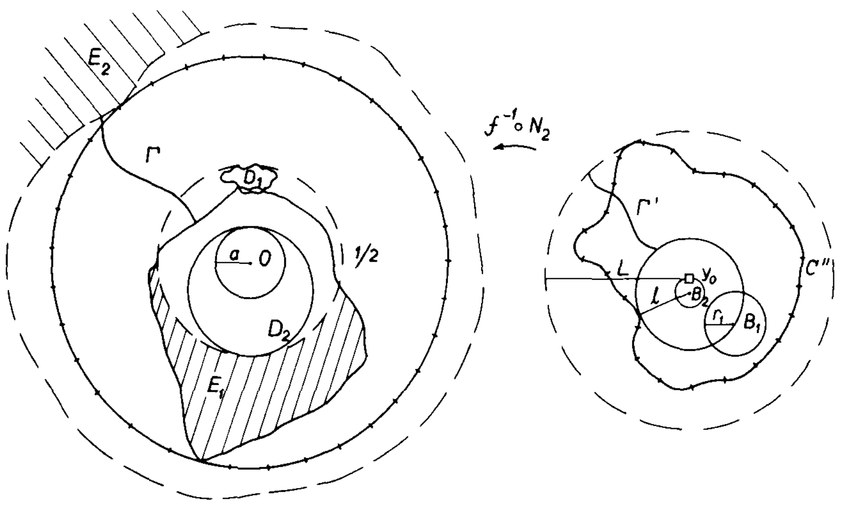


Fig. 2

$$(4.10) \qquad M(\Gamma)/K \leqq M(\Gamma') \leqq M(\Gamma_1').$$

For $M(\Gamma)$ we get an estimate using Lemma 4.7 with $B = B^n(1/2)$, $E_1 = f^{-1} \circ N_2(\bar{B}^n(P_2, l)) \cap CB^n(1/2)$, and $E_2 = f^{-1} \circ N_2(CB^n(y_0, L))$. We obtain

$$(4.11) \qquad M(\Gamma) \geqq M(\Gamma_2) \geqq M(\Gamma_2^*),$$

where $\Gamma_2$ is the family of curves joining $E_1$ to $E_2$ outside $B^n(1/2)$ and $\Gamma_2^*$ joins $E_1^*$ to $E_2^*$ outside $B^n(1/2)$. Since $E_1$ connects $S^{n-1}$ and $S^{n-1}(1/2)$ and $E_2$ connects $S^{n-1}$ and infinity, the cap inequality [6, 10.2] for $E_1^*$ and $E_2^*$ yields

$$(4.12) \qquad M(\Gamma_2^*) \geqq b_n \int_{3/4}^{5/4} s^{-1} ds = b_n \log(5/3).$$

To estimate $M(\Gamma_1')$ we first assume $l/L < 1/2$. Then the family of curves joining $S^{n-1}(y_0, r_2 + l)$ and $S^{n-1}(y_0, L)$ minorizes $\Gamma_1'$ and

$$(4.13) \qquad M(\Gamma_1') \leqq \omega_{n-1}(\log(L/(r_2 + l)))^{1-n} \leqq \omega_{n-1}(\log(L/(2l)))^{1-n}.$$

By the inequalities (4.10)–(4.13)

$$\omega_{n-1}(\log(L/(2l)))^{1-n} \geqq b_n \log(5/3)/K > b_n/(2K),$$

whence $L/l \leqq 2e^c$, where

$$c = (2K\omega_{n-1}/b_n)^{1/(n-1)}.$$

Since $\max\{2, 2e^c\}$ is the latter, we can choose

$$(4.14) \qquad c_2^{-1} = 2e^c.$$

4.15. *The estimation of* $c_3$. We derive an estimate for $c_3$ such that $r_1/l \geqq c_3$. Let $L_1 = \max\{|P_1 - y| : y \in C''\}$. Then $l < L_1$ and

$$(4.16) \qquad r_1/l > r_1/L_1.$$

Let $\Gamma_1'$ be the family of curves joining $\bar{B}_1$ and $S^{n-1}(P_1, L_1)$ outside $B_2$ and $\Gamma_1$ its preimage under $I_3 \circ F$. If $\Gamma'$ joins $\bar{B}_1$ and $S^{n-1}(P_1, L_1)$,

$$(4.17) \qquad M(\Gamma_1)/K \leqq M(\Gamma_1') \leqq M(\Gamma') = \omega_{n-1}(\log(L_1/r_1))^{1-n}$$

and, by (4.16) and (4.17),

$$(4.18) \qquad l/r_1 < \exp((K\omega_{n-1}/M(\Gamma_1))^{1/(n-1)}).$$

The family of curves $\Gamma_1$ joins $\bar{\mathcal{D}}_1$ to the set $M = F^{-1} \circ I_3^{-1}(\bar{B}^n(P_1, L_1))$ in the upper half-space $H$. Let $\tilde{M}$ and $\tilde{D}_1$ be the reflections of $M$ and $\bar{\mathcal{D}}_1$, respectively, in $\partial H$. Denote $D = \bar{\mathcal{D}}_1 \cup \tilde{D}_1$ and let $\Gamma_2$ join $M \cup \tilde{M}$ and $D$ in $\bar{R}^n$. By Lemma 1 of 4.5

$$M(\Gamma_1) = M(\Gamma_2)/2. \tag{4.19}$$

We make a spherical symmetrization of the condenser $(C(M \cup \tilde{M}), D)$ with respect to $\{P + \lambda e_n \mid \lambda \leqq 0\}$. Let $M^*$ be the complement of the symmetrization of $C(M \cup \tilde{M})$ and $D^*$ the symmetrization of $D$. Let $\Gamma_2^*$ join $M^*$ and $D^*$ in $R^n$. Then, due to the result [5, 7.5] about symmetrization (applied after reflections in a sphere with center $P$ and combined with 4.6)

$$M(\Gamma_2) \geqq M(\Gamma_2^*). \tag{4.20}$$

To estimate $M(\Gamma_2^*)$ we use the cap inequality [6, 10.2]. Because the diameter of $\mathscr{D}_1$ is $d$, $\mathscr{D}_1$ contains a point with distance $d/2$ from $y_1$. Denote $x = |P - y_1|$. Then $D^*$ contains at least the segment from $P - (x^2 + (d/2)^2)^{1/2} e_n$ to $P - x e_n$. The set $\partial M$ meets $S^{n-1}(P, s)$ and separates $P$ from $\mathscr{D}_2$ since $S^{n-1}(P_1, L_1)$ separates $B_2$ from infinity, the image of $P$ under $I_3 \circ F$. Therefore $M^*$ contains the segment from $P$ to $P + s e_n$, and

$$M(\Gamma_2^*) \geqq b_n \int t^{-1} dt,$$

with the limits of integration from $x/2$ to

$$x/2 + \min\{(x^2 + (d/2)^2)^{1/2} - x, s\} = -x/2 + (x^2 + (d/2)^2)^{1/2}.$$

By construction, $x = 5R^2/(12r)$, $s/x = 4/5$. Substituting from (3.3) $R/r \leqq 8$ we get, since $R = b + d$, after simplifying

$$M(\Gamma_2^*) > b_n/(50(1 + b/d)^2). \tag{4.21}$$

So, by (4.18)–(4.21) we can choose

$$c_3^{-1} = \exp(100 K \omega_{n-1}(1 + b/d)^2/b_n). \tag{4.22}$$

4.23. *The estimation of* $c_4$. We calculate $c_4$ such that $r_2/l \geqq c_4$. Let $\Gamma'$ be the family of curves joining $B_2$ and $C''$ outside $B_1$, $\Gamma$ its image under $f^{-1} \circ N_2$ and $\Gamma_1'$ the family of curves joining $S^{n-1}(P_2, l)$ and $B_2$. Then

$$M(\Gamma') \leqq M(\Gamma_1') = \omega_{n-1}(\log(l/r_2))^{1-n}.$$

Hence

$$(\log(l/r_2))^{n-1} \leqq \omega_{n-1}/M(\Gamma') \leqq K\omega_{n-1}/M(\Gamma).$$

By construction, since $R = b + d$, the set $D_1$ is in the upper half-space $H$. Let $\Gamma_2$ join $S^{n-1}(a)\backslash H$ and $S^{n-1}\backslash H$ outside $H$. Then

$$M(\Gamma) \geqq M(\Gamma_2) = \omega_{n-1}(\log a^{-1})^{1-n}/2.$$

By (3.4) $1/a \leqq 6$, and we can choose

$$c_4^{-1} = 6^{2K}$$

and

$$\min(c_3, c_4) = c_3. \tag{4.24}$$

## 5. Results

5.1. The construction in lemma 2 [2] gives a formula for the maximal dilatation $K^*$ of the extended mapping $f^*$, and also of $F^*$

$$K^* = K^5 K_2 K_3^3.$$

By (3.5), (4.2) and (4.24) we get with $\varepsilon \to 0$

$$K^* \leqq K^5 3^{3(n-1)}(1 + 9/(c_1c_2c_3))^n. \tag{5.2}$$

5.3. *An estimate for* $\operatorname{mod} \mathcal{A}$. To get an upper bound for $K^*$ which only depends on $n$, $K$ and $\operatorname{mod} \mathcal{A}$ we derive an estimate for $b/d$ in terms of $\operatorname{mod} \mathcal{A}$.

By construction, after a preliminary translation $\mathcal{D}_1$ is contained in the half ball $E = \bar{B}^n(y_2, d) \cap \{x \mid x_n \geqq b\}$, where $y_2 = (0, \cdots, 0, b)$. Let $A_1 = C(E \cup \bar{\mathcal{D}}_2)$. Then $A_1 \subset \mathcal{A}$ and

$$\operatorname{mod} A_1 \leqq \operatorname{mod} \mathcal{A}. \tag{5.4}$$

To estimate $\operatorname{mod} A_1$ consider the cases $d \leqq b/2$ and $d > b/2$. In the first case let $A_2 = C(\bar{B}^n(y_2, d) \cup \bar{\mathcal{D}}_2)$. Then $A_1 \supset A_2$ and

$$\operatorname{mod} A_1 > \operatorname{mod} A_2 = \log(b/d + ((b/d)^2 - 1)^{1/2}) \geqq \log(1 + b/d). \tag{5.5}$$

In the second case $E$ is contained in a ball with radius $(d^2 + (b/2)^2)/b$ and distance $b/2$ from $\mathcal{D}_2$. As in (5.5)

$$\operatorname{mod} A_1 \geqq \log(1 + b/(2d)), \tag{5.6}$$

and, combining (5.4)–(5.6),

$$\operatorname{mod} \mathcal{A} \geqq \log(1 + b/(2d)). \tag{5.7}$$

5.8. *The upper bound.* Substituting in (5.2) the estimates (4.4), (4.14), (4.22), and (5.7) and simplifying yields

$$K^* \leqq C_1(n,K)C_2(n,K)^{\exp(2\bmod \mathcal{A})}(1+K^{2n}\bmod \mathcal{A}^{-2n}),$$

(5.9)
$$C_1(n,K)=K^5 3^{11n},$$
$$C_2(n,K)=\exp(600nK\omega_{n-1}/b_n).$$

From (5.9) we see that the function

(5.10)
$$C_3(n,K)\bmod \mathcal{A}^{-2n},$$
$$C_3(n,K)=C_1(n,K)C_2(n,K)K^{2n}$$

grows as $\bmod \mathcal{A}\to 0$ like the upper bound in (5.9), i.e. their ratio has the limit one as $\bmod \mathcal{A}\to 0$.

5.11. *An example.* Let $c<1$, $B=\{z\in \mathbf{C} \mid c<|z|\leqq 1\}$ and let $f$ be a conformal mapping from $B$ onto $A=\bar{B}^2\cap \mathbf{C}\{z \mid |\operatorname{Re} z|\leqq r, \operatorname{Im} z=0\}$, such that $f$ is symmetric with respect to the coordinate axes and normalized so that $f(1)=1$. We want to find a lower bound for the maximal dilatation $K^*$ of an extremal quasiconformal mapping $f^*\colon \bar{B}^2\to \bar{B}^2$ such that $f^*=f$ on $S^1$.

Let $\Gamma_1$ be the curvefamily joining the sets $C_1=\{e^{i\varphi} \mid |\varphi|\leqq \pi/4\}$ and $C_2=e^{i\pi}C_1$ in $B$ and $\Gamma_1'$ its image under $f$. Then $\Gamma_1'$ joins the arcs $C_1'=f(C_1)$ and $C_2'=f(C_2)$ in $A$, and $M(\Gamma_1)=M(\Gamma_1')$. Let $\Gamma_2$ be the curvefamily joining $C_1$ and $C_2$ in $B^2$ and $\Gamma_2'$ its image under $f^*$. Then $\Gamma_2'$ joins $C_1'$ and $C_2'$ in $B^2$ and, arguing as in the proof of lemma 1, [3], $M(\Gamma_1')=M(\Gamma_2')$. Since $M(\Gamma_2)=1$,

$$K^*\geqq M(\Gamma_2)/M(\Gamma_2')=M(\Gamma_1)^{-1}. \tag{5.12}$$

To estimate $M(\Gamma_1)$ we use the function $\rho$:

$$\rho(z)=2/(\pi|z|)\quad \text{for}\quad \{z \mid \pi/4\leqq|\arg z|\leqq 3\pi/4\},$$
$$\rho(z)=0\quad \text{otherwise}.$$

Then $\rho$ is admissible for $\Gamma_1$ and

$$M(\Gamma_1)\leqq \int_B \rho^2\,dm=4\log(1/c)/\pi=4\bmod B/\pi.$$

By (5.12)

$$K^*\geqq \pi/(4\bmod B).$$

Since $f$ is conformal, $\bmod A=\bmod B$. Hence

$$K^* \geqq \pi/(4 \operatorname{mod} A) \tag{5.13}$$

and our estimate from (5.9) is for $\operatorname{mod} A < 1/2$

$$K^* \leqq C/(\operatorname{mod} A)^4, \tag{5.14}$$

where $C = 2C_1(2,1)C_2(2,1)^e$.

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